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Subdirect Products of Totally Ordered Implicative Algebras

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In [CST] the authors characterized the commutative BCK-algebras which are subdirect products of totally ordered ones by the "identity"[1] $y^{-1}x \vee x^{-1}y = 1$. This was extended in [F] to not necessarily commutative BCK-algebras in part by noting that these are implicative algebras (as defined in [R]) in which the y^{-1} are isotone and commute. However, the unsupported claim in the parenthesis page 384 that this serves to characterize BCK-algebras is false—a counterexample is given in another connection on p. 386 as the positive implicative algebra for which $y^{-1}x = x$ whenever $y \nleq x$. This does not, of course, invalidate the reasoning in [F] but it does lead to the suspicion that the result holds more generally for this subclass of implicative algebras, which properly contains the BCKs. The argument in [F] cannot be taken over verbatim, since at one point use is made of the fact that implicative filters are kernels of morphisms, which holds for BCK but (apparently) not for the larger class. The suspicion is, however, correct and a modified proof, valid for the larger class, is presented below.

We postulate again an "implicative algebra" [R], that is, a set with binary connective, $y^{-1}x$, and constant, 1, such that $y^{-1}x = 1$ is a partial order with 1 as greatest element: this entails $x^{-1}x = x^{-1}1 = 1$. Commutative action of the y^{-1} comes to $y^{-1}x \geq z$ equivalent to $z^{-1}x \geq y$; it entails that $y \to y^{-1}$ is an antitone map to the pointwise ordered selfmaps on X: from $z \leq y \leq (y^{-1}x)^{-1}x$ follows $y^{-1}x \leq z^{-1}x$ for every x. Since $1^{-1}x \geq x$, the z^{-1} are increasing: $z^{-1}x \geq x$. Finally, we require that the z^{-1} be isotone: $x \geq y$ to entail $z^{-1}x \geq z^{-1}y$.

A morphism from a groupoid to an implicative algebra has the preimage of the greatest 1—the "kernel"—include with $y^{-1}x, z^{-1}y$, also $z^{-1}x$, with x every $y^{-1}x$, as well as all $x^{-1}x$. ([R, II, Ex. 1] is in error: condition (f5) need not hold.) Conversely, such a subset ∇ of a groupoid equips it with a preorder for which each element of ∇ is greatest—passing to the

[1] Actually an "identical implication" since $\vee$ is not a composite of operations.

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associated order yields an implicative algebra such that ∇ is in the class of the greatest element. This class consists of the x such that some $y^{-1}x$ and y (hence $x^{-1}y$) belong to ∇. To have commutativity, ∇ should include with $z^{-1}y^{-1}x$, also $y^{-1}z^{-1}x$, and for isotone action, with $y^{-1}x$, also $(z^{-1}y)^{-1}z^{-1}x$. In an implicative filter, this last yields transitivity.

These filters are inductive: if $\{1\}$ is such a filter (which comes to the algebra satisfying all the axioms) then each $x \neq 1$ is excluded by some such filter maximal for excluding x, whose quotient is subdirectly irreducible—i.e., the groupoid is a subdirect product of subdirectly irreducibles.

It may now be shown that for these algebras subdirect irreducibility is equivalent to join-irreducibility of 1. The smallest implicative filter containing p must contain any x for which some $p^{-n}x = 1$, and these x constitute an implicative filter: from $p^{-m}y = p^{-n}y^{-1}x = 1$ thus, $p^{-n}x \geq y$ follows $p^{-(m+n)}x \geq p^{-m}y = 1$. Moreover, $p^{-n}y^{-1}x = 1$ entails $p^{-n}z^{-1}x \geq z^{-1}y$ so isotone action, hence also transitivity, are met. Since an implicative filter is closed under larger element, it is closed under action by inverse of any element. Thus the smallest implicative filter generated by p is the kernel of a surjective morphism for these algebras.

Observe that inverses distribute across inverses of existent joins: since $r^{-1}x \geq p, q$ entails $\geq p \vee q$, $r^{-1}p^{-1}x, r^{-1}q^{-1}x = 1$ entails $r^{-1}(p \vee q)^{-1}x = 1$. Thus if $p^{-m}x = q^{-n}x = 1$ then $(p \vee q)^{-(m+n)}x = 1$, so if 1 is not join-irreducible then the algebra is subdirectly decomposable. Conversely, if 1 is join-irreducible then no pair of proper filters is disjoint.

Total order is equivalent to join-irreducibility of 1 and $y^{-1}x \vee x^{-1}y = 1$. The latter passes from the factors to a subdirect product; hence, it obtains in a subdirect product of totally ordered algebras. Conversely, since it is equivalent to the identity (in binary $y^{-1}x$) $[(x^{-1}y)^{-1}x]^{-1}z \geq (y^{-1}x)^{-1}z$, it passes from an algebra to its (subdirectly irreducible) factors.

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[R] H. RASIOWA, "An Algebraic Approach to Non-classical Logics," Amsterdam, N. Holland, 1974.